IDENTITIES FROM RANDOM MATRICES

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Random Matrices and Gaussian β-Ensembles

A **Random Matrix** is a matrix where all or some of the entries are random variables drawn from a specific probability distribution. In our project, we focus on the limiting behavior of the eigenvalues of the matrix.

Our project focuses on a one-parameter family of models called the **Gaussian β-Ensembles** which have the density function given by

$$p_\beta(\lambda_1,\lambda_2,\dots,\lambda_N)=\frac{1}{Z_{N,\beta}}\prod_{1\le i<j\le N}|\lambda_i-\lambda_j|^\beta\, e^{-\frac{\beta}{4}\sum_{i=1}^N\lambda_i^2},$$

where $Z_{N,\beta}$ is a normalizing constant. When $\beta=1,2,4$, this corresponds to a random matrix with real, complex, and quaternion standard normal elements. These are the classical cases, but we can generalize the model for any $\beta\in\mathbb{R}_{>0}$. In our project, we focus on the situation when $\beta=2n,\ n\in\mathbb{N}$. The histograms of the eigenvalues satisfy a famous result—*Wigner's Semicircle Law.*

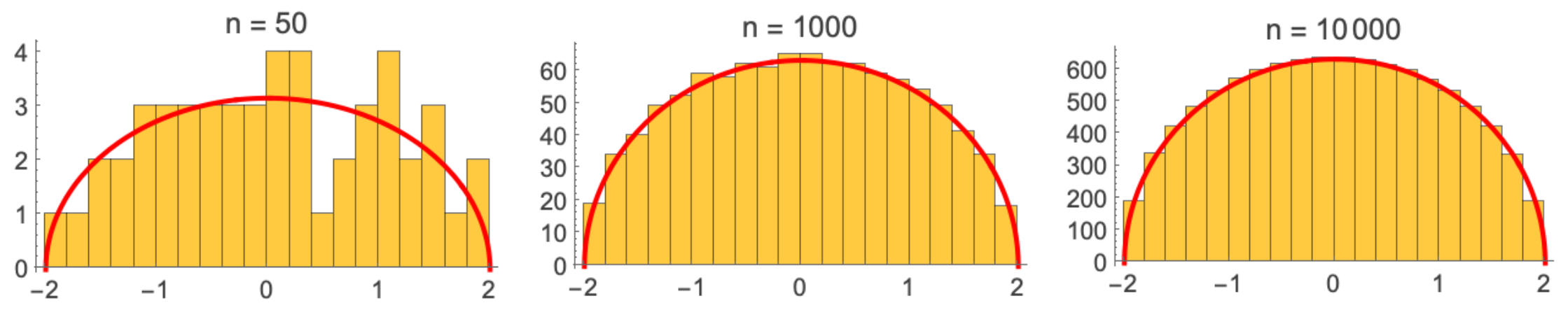


Fig. 1: Simulation of Wigner's Semicircle Law

Point Processes and Correlation Function

As $N\to\infty$, eigenvalues in the interior (bulk) form a limiting **point process**—a random collection of points. For $\beta>0$, this is known as the Sine$_\beta$ process.

The **pair correlation function** $\rho^{(2)}_\beta(0,\lambda)$ gives a measure of the likelihood of finding 2 particles with distance λ apart from each other. Qu and Valkó [2] proved the function is given by an infinite series in terms of a random variable α_λ:

$$\rho^{(2)}_\beta(0,\lambda)=\frac{1}{4\pi^2}+\frac{1}{2\pi^2}\sum_{k=1}^\infty\frac{(-\beta/2)^{\uparrow k}}{(1+\beta/2)^{\uparrow k}}E[\cos(k\alpha_\lambda)]$$

When $\beta=2n$, the *rising factorial* $(x)^{\uparrow k}=x(x+1)(x+2)\cdots(x+k-1)$ truncates the series into a finite sum because $(-n)^{\uparrow n+1}=0$.

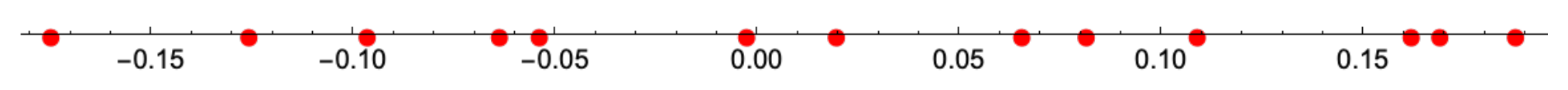


Fig. 2: Simulation of a Point Process for $\beta=1$

Forrester Integral Formula

When $\beta=2n$, Peter Forrester showed there is an integral formula for the 2-point correlation function [1]:

$$\rho^{(2)}_{\beta=2n}(0,\lambda)=\frac{1}{4\pi^2}\cdot\frac{n^{2n}(n!)^3}{(2n)!(3n)!S_{2n}(-1+1/n,-1+1/n,1/n)}\frac{e^{-in\lambda}\lambda^{2n}}{}\times$$

$$\int_{[0,1]^{2n}}\prod_{j=1}^{2n}\left(e^{i\lambda u_j}u_j^{-1+1/n}(1-u_j)^{-1+1/n}\right)\prod_{j<k}|u_j-u_k|^{2/n}\prod_{j=1}^{2n}du_j,$$

where $S_{2n}(-1+1/n,-1+1/n,1/n)$ is defined by

$$\int_{[0,1]^{2n}}\prod_{j=1}^{2n}\left(u_j^{-1+1/n}(1-u_j)^{-1+1/n}\right)\prod_{j<k}|u_j-u_k|^{2/n}\prod_{j=1}^{2n}du_j.$$

By applying the change of variables $u_j=t_j+\frac{1}{2}$, and the Euler's Formula, we showed that the above integral is **indeed real**.

The Vector-Valued ODE System

For $\beta=2n$, we define a vector-valued function $\mathbf{q}(\lambda)=[q_1(\lambda),\dots,q_n(\lambda)]^T$ where $q_k(\lambda)=E[e^{ik\alpha_\lambda}]$. [2] proved that this vector obeys the following first-order linear ODE system:

$$\frac{\beta}{4}\lambda\mathbf{q}'(\lambda)=\left(i\frac{\beta}{4}\lambda\mathbf{B}_n+\mathbf{A}_n\right)\mathbf{q}(\lambda)+\frac{n+1}{2}\mathbf{e}_n,$$

Where $\mathbf{A}_n$ is a tri-diagonal and $\mathbf{B}_n$ is a diagonal matrix with an explicit formula. The correlation function is cleanly recovered via the real part of $\mathbf{q}$:

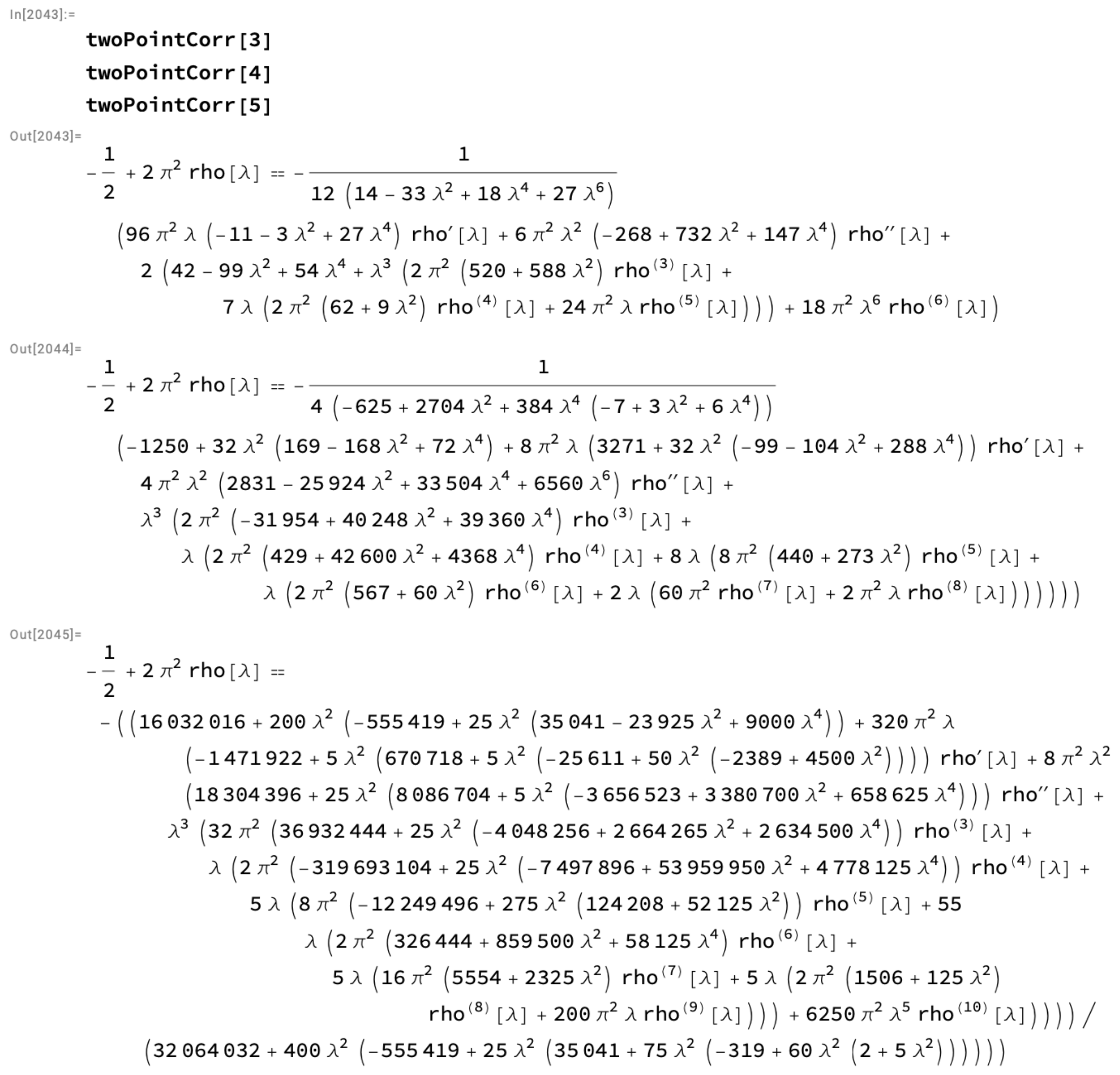
$$\rho^{(2)}_{2n}(0,\lambda)=\frac{1}{4\pi^2}(1+2\mathbf{v}_n^T\Re\mathbf{q}(\lambda)),\tag{1}$$

where $[\mathbf{v}_n]_k=(-1)^k(\binom{2n}{n+k}/\binom{2n}{n}),1\le k\le n.$

Algorithm for ODE System Reduction

We implemented a Mathematica algorithm to automate the reduction of the n-dim system into a single variable $2n$-order ODE with for $h(\lambda):=\Re\mathbf{v_n}^T\mathbf{q}(\lambda)$:

- **Substitution Rules**: Express $\Re q_k'$ and $\Im q_k'$ as linear combinations of $\Re\mathbf{q}$ and $\Im\mathbf{q}$.
- **Recursive Reduction**: Repeatedly differentiate $h(\lambda)=\Re\mathbf{v}_n^T\mathbf{q}(\lambda)$ and substitute lower-order terms to eliminate variables.
- **Reduction to ODE system**: h together with the derivatives $h^{(1)}$, ..., $h^{(2n-1)}$ gives a system of ODE with respect to $\Re\mathbf{q}, Im\mathbf{q}$.
- **Recover the ode for** h: plugging back the solutions of $\Re\mathbf{q}, Im\mathbf{q}$ in terms of $\{h^{(k)}\}_{k=0}^{2n-1}$ to the $2n^{th}$ derivative $h^{(2n)}$ of h yields a $2n$-order ODE for $\rho^{(2)}_{2n}$ for any even β.



```
In[2043]:=
        twoPointCorr[3]
        twoPointCorr[4]
        twoPointCorr[5]
Out[2043]=
          1                                  1
        - ─ + 2 π² rho[λ] == - ─────────────────────────────────────
          2                    12 (14 - 33 λ² + 18 λ⁴ + 27 λ⁶)
          (96 π² λ (-11 - 3 λ² + 27 λ⁴) rho′[λ] + 6 π² λ² (-268 + 732 λ² + 147 λ⁴) rho″[λ] +
            2 (42 - 99 λ² + 54 λ⁴ + λ³ (2 π² (520 + 588 λ²) rho⁽³⁾[λ] +
                  7 λ (2 π² (62 + 9 λ²) rho⁽⁴⁾[λ] + 24 π² λ rho⁽⁵⁾[λ]))) + 18 π² λ⁶ rho⁽⁶⁾[λ])
Out[2044]=
          1                                          1
        - ─ + 2 π² rho[λ] == - ──────────────────────────────────────────────────
          2                    4 (-625 + 2704 λ² + 384 λ⁴ (-7 + 3 λ² + 6 λ⁴))
          (-1250 + 32 λ² (169 - 168 λ² + 72 λ⁴) + 8 π² λ (3271 + 32 λ² (-99 - 104 λ² + 288 λ⁴)) rho′[λ] +
            4 π² λ² (2831 - 25 924 λ² + 33 504 λ⁴ + 6560 λ⁶) rho″[λ] +
            λ³ (2 π² (-31 954 + 40 248 λ² + 39 360 λ⁴) rho⁽³⁾[λ] +
               λ (2 π² (429 + 42 600 λ² + 4368 λ⁴) rho⁽⁴⁾[λ] + 8 λ (8 π² (440 + 273 λ²) rho⁽⁵⁾[λ] +
                     λ (2 π² (567 + 60 λ²) rho⁽⁶⁾[λ] + 2 λ (60 π² rho⁽⁷⁾[λ] + 2 π² λ rho⁽⁸⁾[λ]))))))
Out[2045]=
          1
        - ─ + 2 π² rho[λ] ==
          2
         - ((16 032 016 + 200 λ² (-555 419 + 25 λ² (35 041 - 23 925 λ² + 9000 λ⁴)) + 320 π² λ
                 (-1 471 922 + 5 λ² (670 718 + 5 λ² (-25 611 + 50 λ² (-2389 + 4500 λ²)))) rho′[λ] + 8 π² λ²
                 (18 304 396 + 25 λ² (8 086 704 + 5 λ² (-3 656 523 + 3 380 700 λ² + 658 625 λ⁴))) rho″[λ] +
               λ³ (32 π² (36 932 444 + 25 λ² (-4 048 256 + 2 664 265 λ² + 2 634 500 λ⁴)) rho⁽³⁾[λ] +
                  λ (2 π² (-319 693 104 + 25 λ² (-7 497 896 + 53 959 950 λ² + 4 778 125 λ⁴)) rho⁽⁴⁾[λ] +
                     5 λ (8 π² (-12 249 496 + 275 λ² (124 208 + 52 125 λ²)) rho⁽⁵⁾[λ] + 55
                        λ (2 π² (326 444 + 859 500 λ² + 58 125 λ⁴) rho⁽⁶⁾[λ] +
                           5 λ (16 π² (5554 + 2325 λ²) rho⁽⁷⁾[λ] + 5 λ (2 π² (1506 + 125 λ²)
                                    rho⁽⁸⁾[λ] + 200 π² λ rho⁽⁹⁾[λ]))) + 6250 π² λ⁵ rho⁽¹⁰⁾[λ])))) /
            (32 064 032 + 400 λ² (-555 419 + 25 λ² (35 041 + 75 λ² (-319 + 60 λ² (2 + 5 λ²))))))
```

Fig. 3: Mathematica Program Output

To verify our generated ODEs, we compute the series expansion $\mathbf{q}(\lambda)=\sum s_j\lambda^j$. The coefficients s_k follow the recursion:

$$s_k=i(kI-\frac{4}{\beta}A_n)^{-1}B_ns_{k-1}$$

The numerical consistency between the series and ODE solutions confirms our algebraic derivations.

Classical Cases: $\beta=2$ and $\beta=4$

- When $\beta=2$ ($n=1$): the system collapses to a single *1st-order scalar ODE*:

$$\frac{1}{2}\lambda q_1'(\lambda)=\left(\frac{i}{2}\lambda-1\right)q_1(\lambda)+1,\quad q_1(0)=1$$

Solving this yields the famous **Sine kernel**:

$$\rho^{(2)}_2(0,\lambda)=\frac{1}{4\pi^2}\left[1-\left(\frac{\sin(\lambda/2)}{\lambda/2}\right)^2\right]$$

- When $\beta=4$ ($n=2$):In this case we have a 2^{nd}-order ode by eliminating q_1 yields a *2nd-order ODE* for q_2:

$$\lambda^2q_2''(\lambda)=(-6\lambda+3i\lambda^2)q_2'(\lambda)+(2\lambda^2+8i\lambda-6)q_2(\lambda)+6$$

Solving this correctly recovers the classical **Pfaffian correlation** for the Symplectic ensemble:

$$\rho^{(2)}_4(0,\lambda)=\frac{1}{4\pi^2}\left[1-\operatorname{sinc}^2(\lambda)+\operatorname{sinc}'(\lambda)\int_0^\lambda\operatorname{sinc}(t)dt\right]$$

$\beta=6$ ODE and Its Solution

For $\beta=6$, we derived the 3rd-order complex ODE $L(q_3)=h$ for q_3, where L is 3rd order differential operator with polynomial coefficients, and h is explicit.

Operator Factorization: Using a heuristic search, we find that the 3rd-order differential operator L can be factored into a composition of a 1st-order operator P_1 and a 2nd-order operator P_2:

$$L=P_1\circ P_2$$

By repetitively differentiating the operator L and plugging $\lambda=0$, we find that the boundary condition for the ODE of q_3 are $q_3(0)=1, q_3'(0)=i$, and $q_3''(0)=-\frac{9}{16}$. Using the boundary conditions and solving the first and second order ODEs one after another we get a formula for q_3. Using (1) we get an expression for $\rho_6^{(2)}$ which seems to be new.

Future Work

We aim to further understand the $\beta=6$ case by investigating why the third-order ODE operator for q_3 admits a factorization. We also aim to study the $\beta=8$ case and explore whether a similar operator decomposition can be derived there.

References

References

[1] Peter J. Forrester. *Log-Gases and Random Matrices.* Princeton University Press, 2010.

[2] Yahui Qu and Benedek Valkó. On the pair correlation function of the sine$_\beta$ process. *arXiv preprint arXiv:2509.15446*, 2025.