Identities from random matrices

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Spencer Venancio, Shengqi Qiu, Lingfang Yuan

Faculty mentor: Benedek Valkó, Graduate student mentor: Yahui Qu

Abstract

Random matrix theory provides a rich source of exact identities, but many formulas for local eigenvalue statistics are difficult to derive, verify, and extend beyond the classical cases. We study this problem through the two-point correlation functions of classical beta ensembles. Our project combines probabilistic intuition from point processes, analytic derivations from differential equations, and symbolic computation to understand how these correlation functions arise. As a foundation, we revisit the classical cases $\beta = 2$ and $\beta = 4$, where the associated systems reduce to lower-order differential equations and recover the known sine-kernel and Pfaffian correlation formulas. These cases serve as consistency checks for the differential-equation framework. Building on this, we focus on the less elementary case $\beta = 6$. For this case, the system leads to a third-order complex differential equation with polynomial coefficients, whose direct solution is substantially more difficult than in the classical settings. We derive this equation explicitly, determine its boundary conditions at the origin, and use operator factorization to decompose the third-order problem into a first-order equation followed by a second-order equation. This factorization makes the problem analytically tractable and reveals additional structure hidden in the original system. Solving the resulting equations yields an explicit expression for the corresponding auxiliary function, which can then be substituted back into the correlation formula. Through this process, we obtain a candidate closed-form expression for the two-point correlation function in the $\beta = 6$ case. Together, our work verifies known random-matrix identities in the classical regimes and extends the same method toward a more complicated Gaussian-Beta ensemble, suggesting a systematic route for deriving new correlation formulas from differential-equation structure.

1 Introduction

A **random matrix** is a matrix whose entries are random variables drawn from a specified probability distribution. Random matrix theory began with Wigner's work on the energy levels of heavy nuclei and has since become a central topic in mathematical physics and probability. A fundamental question is: as the size n of the matrix grows, what does the distribution of eigenvalues look like?

A first answer is given by Wigner's Semicircle Law: if the entries are i.i.d. with finite variance, the empirical spectral distribution of the rescaled eigenvalues converges weakly to the semicircle distribution $\frac{1}{2\pi}\sqrt{4-x^2}\,\mathbf{1}_{[-2,2]}(x)$ as $n \to \infty$. Figure 1 illustrates this convergence via simulation.

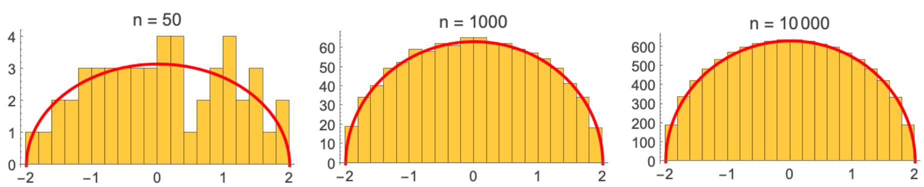

Figure 1: Simulation of Wigner's Semicircle Law for a 500×500 GUE matrix.

Beyond the global shape, one can zoom in on the bulk of the semicircle and ask about the *local* statistics of eigenvalues — how they repel one another, and how the repulsion depends on the symmetry class of the matrix. As $n \to \infty$ the rescaled local eigenvalue statistics converge to a limiting point process called the Sine_β process, illustrated in Figure 2.

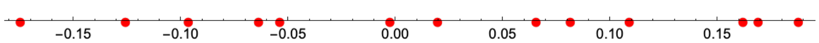

Figure 2: Simulation of a point process for $\beta = 1$, showing level repulsion.

The central object of our project is the **pair correlation function** $\rho_\beta^{(2)}(0, \lambda)$ of the Sine_β process, which measures the likelihood of finding two eigenvalues a distance λ apart. For the classical values $\beta = 1, 2, 4$ this function is explicitly known; for general β it is much harder to compute. Our goal is to make progress on $\beta = 6$, using a differential-equation approach developed in [2].

The project is organized as follows. In Section 2 we review Gaussian β-ensembles and the Selberg-type integral formula due to Forrester [1]. In Section 3 we define correlation functions and introduce the series representation for $\rho_\beta^{(2)}$. Section 4 presents the vector-valued ODE system that governs the building blocks of the correlation function. Sections 5 and 6 treat the classical cases $\beta = 2, 4$ and the Mathematica implementation used to generate ODEs.

Section 7 presents our main new results for $\beta = 6$. We conclude with future directions in Section 8.

2 Background: Gaussian β-Ensembles

2.1 Classical matrix models

Let A be an $N \times N$ matrix with i.i.d. standard Gaussian entries over $\mathbb{R}$, $\mathbb{C}$, or the quaternions $\mathbb{H}$. The symmetrization

$$X = \frac{A + A^{\intercal}}{\sqrt{2}}$$

(using the appropriate conjugate transpose) gives the Gaussian Orthogonal Ensemble (GOE, $\beta = 1$), Gaussian Unitary Ensemble (GUE, $\beta = 2$), and Gaussian Symplectic Ensemble (GSE, $\beta = 4$).

Theorem 1 ([1]). *The joint probability density function of the eigenvalues* $(\lambda_1, \ldots, \lambda_N)$ *of the above ensembles is*

$$p_\beta(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_{N,\beta}} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{\beta}{4} \sum_{i=1}^{N} \lambda_i^2 \right), \tag{1}$$

where $\beta = 1, 2, 4$ *and* $Z_{N,\beta}$ *is a normalizing constant.*

The product $\prod_{i<j} |\lambda_i - \lambda_j|$ is the absolute value of the *Vandermonde determinant*, and the $|\lambda_i - \lambda_j|^\beta$ factor encodes eigenvalue repulsion: larger β means stronger repulsion.

2.2 The Gaussian β-ensemble

The density (1) makes sense for any $\beta > 0$, not just $\beta = 1, 2, 4$. The *Gaussian β-ensemble* (GβE) is the family of probability measures on $\mathbb{R}^N$ with this joint density. The classical ensembles are the special cases $\beta = 1, 2, 4$, but the limit as $N \to \infty$ — the Sine_β process — is defined for all $\beta > 0$.

In this project we focus on *even* values $\beta = 2n$, $n \in \mathbb{N}$. This restriction is natural because the ODE system that governs the correlation function truncates to a finite closed system only when β is a positive even integer.

2.3 The Forrester integral formula

For $\beta = 2n$, Peter Forrester derived an explicit integral formula for the pair correlation function [1]:

$$\rho^{(2)}_{\beta=2n}(0,\lambda) = \frac{1}{4\pi^2}\cdot\frac{n^{2n}(n!)^3}{(2n)!\,(3n)!}\cdot\frac{e^{-in\lambda}\lambda^{2n}}{S_{2n}(-1+1/n,-1+1/n,1/n)}\times$$
$$\int_{[0,1]^{2n}}\prod_{j=1}^{2n}\left(e^{i\lambda u_j}u_j^{-1+1/n}(1-u_j)^{-1+1/n}\right)\prod_{j<k}|u_j-u_k|^{2/n}\prod_{j=1}^{2n}du_j, \tag{2}$$

where $S_{2n}(-1+1/n,-1+1/n,1/n)$ is a Selberg-type normalizing constant defined by

$$S_{2n}(-1+1/n,-1+1/n,1/n) := \int_{[0,1]^{2n}}\prod_{j=1}^{2n}\left(u_j^{-1+1/n}(1-u_j)^{-1+1/n}\right)\prod_{j<k}|u_j-u_k|^{2/n}\prod_{j=1}^{2n}du_j,$$

and the general Selberg integral evaluates as

$$S_n(\alpha,\beta,\gamma) := \int_{[0,1]^n}\prod_{i=1}^{n}t_i^{\alpha-1}(1-t_i)^{\beta-1}\prod_{j<k}|t_j-t_k|^{2\gamma}\,dt_1\cdots dt_n = \prod_{j=0}^{n-1}\frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(1+\gamma)}.$$

Despite the complex-valued integrand, the correlation function $\rho^{(2)}_{\beta=2n}(0,\lambda)$ is real-valued. The key observation is the following.

Proposition 2 (The Forrester integral is real)**.** *The integral in* (2) *is real.*

Proof. Apply the change of variables $u_j = t_j + \frac{1}{2}$. Then $e^{i\lambda u_j} = e^{i\lambda/2}\,e^{i\lambda t_j}$, so the product $e^{-in\lambda}\prod_j e^{i\lambda u_j} = \prod_j e^{i\lambda t_j}$. The full integrand (after absorbing $e^{-in\lambda}$) becomes

$$\prod_{j=1}^{2n}\big(\cos(\lambda t_j)+i\sin(\lambda t_j)\big)\cdot\prod_{j=1}^{2n}\left(\tfrac{1}{4}-t_j^2\right)^{-1+1/n}\cdot\prod_{j<k}|t_j-t_k|^{2/n},$$

integrated over $[-1/2,1/2]^{2n}$. The weight $\prod_j(\frac{1}{4}-t_j^2)^{-1+1/n}\prod_{j<k}|t_j-t_k|^{2/n}$ is an even, symmetric function of $(t_1,\ldots,t_{2n})$. The imaginary part of the integrand contains the factor $\sin(\lambda\sum_j t_j)$, which is odd under $(t_1,\ldots,t_{2n})\mapsto(-t_1,\ldots,-t_{2n})$. Since the domain is symmetric, the imaginary part integrates to zero, so the integral is real. □

3 Point Processes and Correlation Functions

3.1 Point processes

A **point process** is a random locally finite collection of points on $\mathbb{R}$ (or $\mathbb{R}^d$). The eigenvalues of a random matrix form a point process of N points; as $N\to\infty$ and we zoom into the bulk of the semicircle, these converge to the Sine_β process.

3.2 Correlation functions

Definition 3. The k-point correlation function (or intensity function) $\rho_k(x_1, \ldots, x_k)$ of a point process is the function satisfying

$$\mathbb{P}\big(X_l \in (x_l, x_l + \varepsilon) \text{ for } l = 1, \ldots, k\big) \approx \varepsilon^k \, \rho_k(x_1, \ldots, x_k)$$

as $\varepsilon \to 0$, where the $x_1, \ldots, x_k$ are distinct.

For the Poisson process with rate λ, one shows that $\rho_k(x_1, \ldots, x_k) = \lambda^k$ for all k. The Sine_β process is far from Poisson; it exhibits repulsion between nearby points, reflected in the fact that $\rho_2(0, \lambda) \to 0$ as $\lambda \to 0$.

Because the Sine_β process is translation invariant, the pair correlation function satisfies $\rho^{(2)}_\beta(x, y) = \rho^{(2)}_\beta(0, x - y)$. We therefore write $\rho^{(2)}_\beta(0, \lambda)$ for the pair correlation as a function of the signed distance λ.

The normalization is fixed by requiring the one-point intensity to equal the bulk density of the semicircle at the zooming point, which is $\rho^{(1)} = \frac{1}{2\pi}$ at the origin.

3.3 Series representation

Qu and Valkó [2] proved that for $\beta > 0$, the pair correlation function admits the series representation

$$\rho^{(2)}_\beta(0, \lambda) = \frac{1}{4\pi^2} + \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{(-\beta/2)^{\uparrow k}}{(1 + \beta/2)^{\uparrow k}} E\big[\cos(k\alpha_\lambda)\big], \tag{3}$$

where $(x)^{\uparrow k} = x(x+1)\cdots(x+k-1)$ is the rising factorial and α_λ is a certain $\mathbb{R}$-valued random variable.

A key simplification occurs when $\beta = 2n$ is a positive even integer: the rising factorial $(-n)^{\uparrow k}$ vanishes for $k \geq n+1$, so the infinite series (3) truncates to a finite sum:

$$\rho^{(2)}_{\beta=2n}(0, \lambda) = \frac{1}{4\pi^2} + \frac{1}{2\pi^2} \sum_{k=1}^{n} \frac{(-n)^{\uparrow k}}{(1+n)^{\uparrow k}} E\big[\cos(k\alpha_\lambda)\big]. \tag{4}$$

The coefficients simplify nicely. One can check that

$$\frac{(-n)^{\uparrow k}}{(1+n)^{\uparrow k}} = (-1)^k \frac{n! \, n!}{(n-k)! \, (n+k)!} = (-1)^k \frac{\binom{2n}{n-k}}{\binom{2n}{n}}.$$

Defining $[\mathbf{v}_n]_k = (-1)^k \binom{2n}{n+k} / \binom{2n}{n}$ for $1 \leq k \leq n$, we can write

$$\rho^{(2)}_{\beta=2n}(0, \lambda) = \frac{1}{4\pi^2}\big(1 + 2\, \mathbf{v}_n^T \Re \mathbf{q}(\lambda)\big), \tag{5}$$

where $\mathbf{q}(\lambda) = [q_1(\lambda), \ldots, q_n(\lambda)]^T$ with $q_k(\lambda) = E[e^{ik\alpha_\lambda}]$, and $\Re$ denotes the componentwise real part.

4 The Vector-Valued ODE System

4.1 Setting up the system

Qu and Valkó [2] showed that the vector $\mathbf{q}(\lambda)$ satisfies the first-order linear ODE system

$$\frac{\beta}{4}\lambda\,\mathbf{q}'(\lambda) = \Big(i\tfrac{\beta}{4}\lambda\,\mathbf{B}_n + \mathbf{A}_n\Big)\mathbf{q}(\lambda) + \tfrac{n+1}{2}\,\mathbf{e}_n, \qquad \mathbf{q}(0) = \mathbf{f}_n, \tag{6}$$

where $\mathbf{e}_n = [1, 0, \dots, 0]^T$, $\mathbf{f}_n = [1, 1, \dots, 1]^T \in \mathbb{R}^n$, and $\mathbf{A}_n, \mathbf{B}_n$ are $n \times n$ matrices with entries

$$[\mathbf{A}_n]_{k,k} = -k^2, \quad [\mathbf{A}_n]_{k,k-1} = \tfrac{1}{2}k(k+n), \quad [\mathbf{A}_n]_{k,k+1} = \tfrac{1}{2}k(k-n), \quad [\mathbf{B}_n]_{k,k} = k.$$

(All other entries of $\mathbf{A}_n$ and $\mathbf{B}_n$ are zero.) When $\beta = 2n$, the system (6) is a closed n-dimensional system; the initial condition $q_k(0) = 1$ for all k follows from $E[e^0] = 1$.

4.2 Power series solution

To verify solutions and generate Taylor series, we substitute $\mathbf{q}(\lambda) = \sum_{j=0}^{\infty} \mathbf{s}_j \lambda^j$ into (6) and match powers of λ. This gives the recursion

$$\mathbf{s}_0 = \mathbf{f}_n, \qquad \mathbf{s}_k = i\Big(k\mathbf{I} - \tfrac{4}{\beta}\mathbf{A}_n\Big)^{-1}\mathbf{B}_n\,\mathbf{s}_{k-1}, \quad k \geq 1. \tag{7}$$

Because $\mathbf{A}_n$ is a polynomial function of k, the matrix $(k\mathbf{I} - \frac{4}{\beta}\mathbf{A}_n)$ is invertible for all $k \geq 1$. Every other coefficient $\mathbf{s}_k$ is purely imaginary (odd k) or real (even k), so

$$\Re\mathbf{q}(\lambda) = \sum_{j=0}^{\infty} \mathbf{s}_{2j}\lambda^{2j}.$$

5 Classical Cases: $\beta = 2$ and $\beta = 4$

5.1 The case $\beta = 2$ $(n = 1)$

For $\beta = 2$, the system (6) reduces to the scalar ODE

$$\frac{\lambda}{2}q_1'(\lambda) = 1 - q_1(\lambda) + \frac{i\lambda}{2}q_1(\lambda), \qquad q_1(0) = 1. \tag{8}$$

Rewriting in standard form $q_1' + P(\lambda)q_1 = Q(\lambda)$ with $P(\lambda) = \frac{2}{\lambda} - i$ and $Q(\lambda) = \frac{2}{\lambda}$, we use the integrating factor $\mu(\lambda) = \lambda^2 e^{-i\lambda}$:

$$\frac{d}{d\lambda}\big(\lambda^2 e^{-i\lambda} q_1(\lambda)\big) = 2\lambda e^{-i\lambda}.$$

Integrating by parts and imposing $q_1(0) = 1$ to fix the constant of integration gives

$$q_1(\lambda) = \frac{2(1 + i\lambda - e^{i\lambda})}{\lambda^2}. \tag{9}$$

Proposition 4. *The formula* (9) *recovers the classical sine kernel:*

$$\rho^{(2)}_{\beta=2}(0,\lambda) = \frac{1}{4\pi^2}\left(1 - \frac{\sin^2(\lambda/2)}{(\lambda/2)^2}\right).$$

Proof. Since $\mathbf{v}_1 = -\frac{1}{2}$ and $\rho^{(2)}_{\beta=2}(0,\lambda) = \frac{1}{4\pi^2}(1 + 2\mathbf{v}_1^T \Re q_1(\lambda)) = \frac{1}{4\pi^2}(1 - \Re q_1(\lambda))$, we compute

$$\Re q_1(\lambda) = \Re \frac{2(1 + i\lambda - e^{i\lambda})}{\lambda^2} = \frac{2(1-\cos\lambda)}{\lambda^2} = \frac{4\sin^2(\lambda/2)}{\lambda^2} = \frac{\sin^2(\lambda/2)}{(\lambda/2)^2}. \qquad \square$$

5.2 The case $\beta = 4$ ($n = 2$)

For $\beta = 4$, the system has $n = 2$ components. Writing it in standard form, we get

$$\begin{aligned} q_1' &= \left(i - \tfrac{1}{\lambda}\right) q_1 - \tfrac{1}{2\lambda} q_2 + \tfrac{3}{2\lambda}, \\ q_2' &= \tfrac{4}{\lambda} q_1 + \left(2i - \tfrac{4}{\lambda}\right) q_2. \end{aligned}$$

To solve the homogeneous system we apply a gauge transformation. Setting gauge factors $u = e^{i\lambda}/\lambda$ and $v = e^{2i\lambda}/\lambda^4$ and writing $q_1 = fu$, $q_2 = gv$, the system decouples to a second-order scalar ODE for g:

$$g'' + \left(i - \frac{2}{\lambda}\right) g' + \frac{2}{\lambda^2} g = 0.$$

The two independent solutions involve the non-elementary integral $I(\lambda) = \int \frac{e^{i\lambda}}{\lambda^2}\, d\lambda$. Using variation of parameters with the non-homogeneous term $\mathbf{g} = (3/(2\lambda), 0)^T$, the final solution for q_2 is

$$q_2(\lambda) = \frac{3}{\lambda^2}\left[e^{i\lambda}\frac{\sin\lambda}{\lambda} - ie^{i\lambda}\,\mathrm{Si}(\lambda) - 1\right],$$

where $\mathrm{Si}(\lambda) = \int_0^\lambda \frac{\sin t}{t}\, dt$ is the sine integral.

Proposition 5. *The solution for* $\beta = 4$ *recovers the Pfaffian formula:*

$$\rho^{(2)}_{\beta=4}(0,\lambda) = \frac{1}{4\pi^2}\left[1 - \left(\frac{\sin\lambda}{\lambda}\right)^2 + \frac{d}{d\lambda}\left(\frac{\sin\lambda}{\lambda}\right)\int_0^\lambda \frac{\sin t}{t}\, dt\right].$$

Remark 6. Near $\lambda = 0$, the correlation function behaves as $\rho^{(2)}_{\beta=4}(0,\lambda) \approx \frac{\lambda^4}{540\pi^2}$, which can be confirmed via Taylor expansion. This λ^4 vanishing reflects stronger level repulsion at $\beta = 4$ compared to the λ^2 behavior at $\beta = 2$.

Figure 3 shows the Taylor series approximation of $\rho^{(2)}_4$ near $\lambda = 0$.

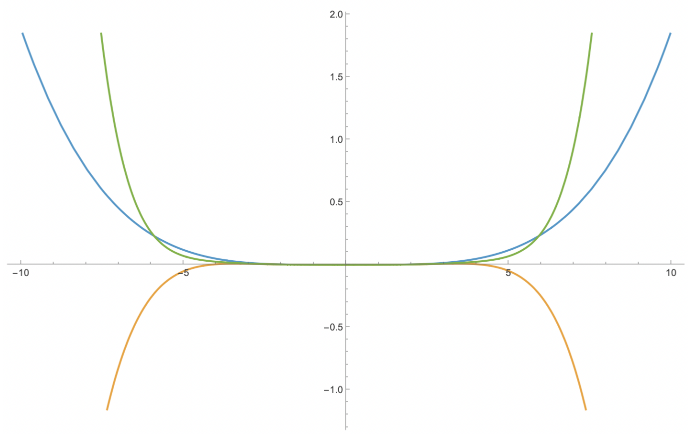

Figure 3: Taylor series approximation of $\rho_4^{(2)}(0,\lambda)$.

5.3 Verification via Forrester's formula ($n = 1$)

As an independent check for $\beta = 2$, we verify the $n = 1$ case of (2). The relevant Selberg constant is $S_2(0,0,1) = \int_0^1 \int_0^1 (u_1 - u_2)^2 \, du_1 \, du_2 = 1/6$. Via the change of variables $x = u_1 + u_2$, $y = u_1 - u_2$, one computes

$$\int_0^1 \int_0^1 e^{i\lambda(u_1+u_2)}(u_1 - u_2)^2 \, du_1 \, du_2 = \frac{2e^{i\lambda}(\lambda^2 + 2\cos\lambda - 2)}{\lambda^4}.$$

Substituting into (2) with $n = 1$ and simplifying using $1 - \cos\lambda = 2\sin^2(\lambda/2)$ recovers

$$\rho^{(2)}_{\beta=2}(0,\lambda) = \frac{1}{4\pi^2}\left(1 - \frac{\sin^2(\lambda/2)}{(\lambda/2)^2}\right),$$

confirming consistency between the two approaches.

6 Algorithmic Reduction to a Single ODE

For general even $\beta = 2n$, we implemented a Mathematica algorithm that automatically reduces the n-dimensional system (6) to a single scalar ODE for the function $h(\lambda) := \Re \mathbf{v}_n^T \mathbf{q}(\lambda)$. By (5), recovering h is equivalent to recovering $\rho^{(2)}_{2n}$.

The algorithm proceeds in three stages:

1. **Substitution rules.** From the ODE system, extract linear relations expressing $(\Re q_k)'$ and $(\Im q_k)'$ as linear combinations of the $2n$ real functions $\{\Re q_k, \Im q_k\}_{k=1}^n$.

2. **Recursive differentiation.** Starting from $h = \Re \mathbf{v}_n^T \mathbf{q}$, differentiate repeatedly and substitute the rules from step 1. This expresses $h, h', h'', \ldots, h^{(2n-1)}$ as linear combinations of $\{\Re q_k, \Im q_k\}_{k=1}^n$, giving a $2n \times 2n$ linear system.

3. **Back-substitution.** Solve the linear system from step 2 for $\{\Re q_k, \Im q_k\}$ in terms of $\{h^{(j)}\}_{j=0}^{2n-1}$ using Mathematica's `LinearSolve`. Substituting into the expression for $h^{(2n)}$ yields the desired $2n$-th order ODE for h, and hence for $\rho_{2n}^{(2)}$.

Figure 4 shows a sample output of the program.

```
In[2043]:=
        twoPointCorr[3]
        twoPointCorr[4]
        twoPointCorr[5]
Out[2043]=
  -1/2 + 2 π^2 rho[λ] == -1/(12 (14 - 33 λ^2 + 18 λ^4 + 27 λ^6))
    (96 π^2 λ (-11 - 3 λ^2 + 27 λ^4) rho'[λ] + 6 π^2 λ^2 (-268 + 732 λ^2 + 147 λ^4) rho''[λ] +
      2 (42 - 99 λ^2 + 54 λ^4 + λ^3 (2 π^2 (520 + 588 λ^2) rho^(3)[λ] +
            7 λ (2 π^2 (62 + 9 λ^2) rho^(4)[λ] + 24 π^2 λ rho^(5)[λ]))) + 18 π^2 λ^6 rho^(6)[λ])
Out[2044]=
  -1/2 + 2 π^2 rho[λ] == -1/(4 (-625 + 2704 λ^2 + 384 λ^4 (-7 + 3 λ^2 + 6 λ^4)))
    (-1250 + 32 λ^2 (169 - 168 λ^2 + 72 λ^4) + 8 π^2 λ (3271 + 32 λ^2 (-99 - 104 λ^2 + 288 λ^4)) rho'[λ] +
      4 π^2 λ^2 (2831 - 25 924 λ^2 + 33 504 λ^4 + 6560 λ^6) rho''[λ] +
      λ^3 (2 π^2 (-31 954 + 40 248 λ^2 + 39 360 λ^4) rho^(3)[λ] +
         λ (2 π^2 (429 + 42 600 λ^2 + 4368 λ^4) rho^(4)[λ] + 8 λ (8 π^2 (440 + 273 λ^2) rho^(5)[λ] +
               λ (2 π^2 (567 + 60 λ^2) rho^(6)[λ] + 2 λ (60 π^2 rho^(7)[λ] + 2 π^2 λ rho^(8)[λ]))))))
Out[2045]=
  -1/2 + 2 π^2 rho[λ] ==
   -((16 032 016 + 200 λ^2 (-555 419 + 25 λ^2 (35 041 - 23 925 λ^2 + 9000 λ^4)) + 320 π^2 λ
          (-1 471 922 + 5 λ^2 (670 718 + 5 λ^2 (-25 611 + 50 λ^2 (-2389 + 4500 λ^2)))) rho'[λ] + 8 π^2 λ^2
          (18 304 396 + 25 λ^2 (8 086 704 + 5 λ^2 (-3 656 523 + 3 380 700 λ^2 + 658 625 λ^4))) rho''[λ] +
        λ^3 (32 π^2 (36 932 444 + 25 λ^2 (-4 048 256 + 2 664 265 λ^2 + 2 634 500 λ^4)) rho^(3)[λ] +
           λ (2 π^2 (-319 693 104 + 25 λ^2 (-7 497 896 + 53 959 950 λ^2 + 4 778 125 λ^4)) rho^(4)[λ] +
              5 λ (8 π^2 (-12 249 496 + 275 λ^2 (124 208 + 52 125 λ^2)) rho^(5)[λ] + 55
                 λ (2 π^2 (326 444 + 859 500 λ^2 + 58 125 λ^4) rho^(6)[λ] +
                    5 λ (16 π^2 (5554 + 2325 λ^2) rho^(7)[λ] + 5 λ (2 π^2 (1506 + 125 λ^2)
                             rho^(8)[λ] + 200 π^2 λ rho^(9)[λ]))) + 6250 π^2 λ^5 rho^(10)[λ])))) /
     (32 064 032 + 400 λ^2 (-555 419 + 25 λ^2 (35 041 + 75 λ^2 (-319 + 60 λ^2 (2 + 5 λ^2))))))
```

Figure 4: Mathematica program output generating the ODE for $\rho_{\beta=2n}^{(2)}$.

For $\beta = 6$ ($n = 3$), the algorithm produces the explicit sixth-order ODE

$$
\begin{aligned}
\frac{81\lambda^6}{\pi^2} = {} & 9\lambda^6\,\partial_\lambda^6\rho_6 + 168\lambda^5\,\partial_\lambda^5\rho_6 + (126\lambda^6 + 868\lambda^4)\,\partial_\lambda^4\rho_6 \\
& + (1176\lambda^5 + 1040\lambda^3)\,\partial_\lambda^3\rho_6 + (441\lambda^6 + 2196\lambda^4 - 804\lambda^2)\,\partial_\lambda^2\rho_6 \\
& + (1296\lambda^5 - 144\lambda^3 - 528\lambda)\,\partial_\lambda\rho_6 + (324\lambda^6 + 216\lambda^4 - 396\lambda^2 + 168)\,\rho_6,
\end{aligned}
$$

which we verified against the power series recursion (7).

7 The $\beta = 6$ Case

7.1 The ODE for q_3

For $\beta = 6$ ($n = 3$), after eliminating q_1 and q_2 from the system (6), one obtains the third-order ODE for q_3:

$$\begin{aligned} &3i\lambda^3 q_3'''(\lambda) + \left(18\lambda^3 + 37i\lambda^2\right) q_3''(\lambda) + \left(-33i\lambda^3 + 138\lambda^2 + 115i\lambda\right) q_3'(\lambda) \\ &\quad + \left(-18\lambda^3 - 117i\lambda^2 + 195\lambda + 80i\right) q_3(\lambda) = 80i. \end{aligned} \tag{10}$$

7.2 Boundary conditions

Setting $\lambda = 0$ in (10) immediately gives $q_3(0) = 1$. Differentiating and setting $\lambda = 0$ gives

$$115i\, q_3'(0) + 195\, q_3(0) + 80i = 0 \implies q_3'(0) = i.$$

Differentiating again and setting $\lambda = 0$ yields $q_3''(0) = -9/8$.

7.3 Change of variables and operator factorization

The asymptotic behavior of (10) as $\lambda \to \infty$ suggests that the highest-order terms dominate with characteristic exponents $r \in \{i, 2i, 3i\}$, motivating the substitution $q_3 = e^{3i\lambda}\lambda^{\alpha} v$.

Substituting this ansatz into (10) and requiring the constant term to vanish yields the algebraic equation

$$3\alpha^3 + 28\alpha^2 + 84\alpha + 80 = (\alpha + 2)(\alpha + 4)(3\alpha + 10) = 0,$$

giving possible values $\alpha \in \{-2, -4, -\frac{10}{3}\}$.

Taking $\alpha = -2$, the equation becomes

$$3i\lambda^2 v''' + \left(-9\lambda^2 + 19i\lambda\right) v'' + \left(-6i\lambda^2 - 48\lambda + 21i\right) v' + \left(-24i\lambda - 36\right) v = 80i\lambda e^{-3i\lambda}.$$

Theorem 7 (Operator factorization)**.** *The homogeneous part of the above ODE factors as*

$$P_1 \circ P_2, \tag{11}$$

where

$$\begin{aligned} P_1 &= \lambda D_\lambda + 3, \\ P_2 &= 3i\lambda D_\lambda^2 + (-9\lambda + 7i) D_\lambda + (-6i\lambda - 12). \end{aligned}$$

7.4 Solving the factored system

The factorization (11) converts the problem into two sequential lower-order ODEs. Setting $w = P_2[v]$, the equation $P_1[w] = 80i\lambda e^{-3i\lambda}$ is first-order and yields

$$w = C_0\lambda^{-3} + e^{-3i\lambda}\left(-\tfrac{80}{3} + \tfrac{80i}{3\lambda} + \tfrac{160}{9\lambda^2} - \tfrac{160i}{27\lambda^3}\right).$$

One then solves the second-order equation $P_2[v] = w$ via Mathematica. The two independent solutions of the homogeneous part of $P_2[v] = 0$ are expressed in terms of the Kummer confluent hypergeometric functions U and L:

$$U_\lambda := U\left(\tfrac{2}{3}, \tfrac{7}{3}, i\lambda\right), \qquad L_\lambda := L_{-2/3}^{4/3}(i\lambda),$$

and the particular solution is obtained by variation of parameters.

The full solution for v is

$$v(\lambda) = e^{-2i\lambda}\left(U_\lambda \int_1^\lambda h_L(s)\,ds + L_\lambda \int_1^\lambda h_U(t)\,dt\right) + c_1 e^{-2i\lambda}U_\lambda + c_2 e^{-2i\lambda}L_\lambda,$$

where the kernels h_L, h_U are explicit rational-exponential functions of the integrand, and the constants c_1, c_2 are determined by the boundary conditions $q_3(0) = 1$, $q_3'(0) = i$, $q_3''(0) = -9/8$.

Substituting $q_3 = e^{3i\lambda}\lambda^{-2}v$ back into (5) with $\mathbf{v}_3 = [-3/4, 3/10, -1/20]$ then gives a candidate closed-form expression for $\rho^{(2)}_{\beta=6}(0,\lambda)$. This formula appears to be new.

7.5 Alternative Decomposition of the ODE for q_3

The (10) is the same as

$$18i\lambda^3 q_3''(\lambda) - 3\lambda^3 q_3^{(3)}(\lambda) - 37\lambda^2 q_3''(\lambda) + \left(33\lambda^3 + 138i\lambda^2 - 115\lambda\right) q_3'(\lambda) + \left(-18i\lambda^3 + 117\lambda^2 + 195i\lambda - 80\right) q_3(\lambda) + 80 = 0$$

We find, through a Mathematica program which searches through possible ansatz that

$$p_1 = -\lambda D(\lambda) + (-4 + 3i\lambda)$$
$$p_2 = 3\lambda^2 D^2(\lambda) + \left(19\lambda - 9i\lambda^2\right) D(\lambda) + \left(-6\lambda^2 - 27i\lambda + 20\right)$$

satisfy

$$(p_1 \circ p_2)(q(\lambda)) = \lambda\Big(\left(33\lambda^2 + 138i\lambda - 115\right) q_3'(\lambda) + \lambda\left(-3\lambda q_3^{(3)}(\lambda) + (-37 + 18i\lambda) q_3''(\lambda)\right)\Big) + \left(-18i\lambda^3 + 117\lambda^2 + 195i\lambda - 80\right) q_3(\lambda).$$

This illustrates that the ODE for q_3 is decomposable, however, it is hard to recover a solution from the direct decomposition above, since the involved second-order differential equation is not of a clear special form.

8 Conclusion and Future Work

We have studied the pair correlation function of the Sine_β process using a differential-equation approach. Our contributions are:

- Explicit derivation and solution of the ODE system for $\beta = 2$ and $\beta = 4$, recovering the classical sine-kernel and Pfaffian formulas.
- A Mathematica algorithm that automatically generates the $2n$-th order ODE for $\rho^{(2)}_{\beta=2n}$ for any n, verified against the power series recursion.
- A proof that Forrester's integral formula is real (Proposition 2) and a direct verification of the $n = 1$ case.
- For $\beta = 6$: derivation of the third-order ODE for q_3, determination of boundary conditions, discovery of an operator factorization $L = P_1 \circ P_2$, and an explicit (though intricate) closed-form expression for $\rho^{(2)}_6$.

Several natural directions remain open:

Question 8. *Why does the third-order differential operator for q_3 in the $\beta = 6$ case admit a factorization into operators of orders 1 and 2? Is there a structural explanation, for instance related to the representation theory of $\mathfrak{sl}_2$ or to properties of the Selberg integral?*

Question 9. *Does the same factorization pattern persist for $\beta = 8$ ($n = 4$)? More generally, is there a systematic way to decompose the n-th order ODE for q_n for any n?*

Question 10. *Can one find a Forrester-type integral formula for the vector $\mathbf{q}(\lambda)$ itself, rather than just for $\rho^{(2)}_{\beta=2n}$? For $\beta = 4$ the explicit formula for q_2 might suggest an ansatz.*

Answering these questions would bring us closer to a unified analytic description of the Sine_β process for all positive even integers β.

References

[1] Peter J. Forrester. *Log-Gases and Random Matrices.* Princeton University Press, 2010.

[2] Yahui Qu and Benedek Valkó. On the pair correlation function of the sine_β process. *arXiv preprint arXiv:2509.15446*, 2025.